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# Jacobsons Lemma fails for nil-clean $2 \times 2$ integral matrices

**Grigore Călugăreanu and Horia F. Pop**

**Abstract**

We show that for two $2 \times 2$ integral matrices A, B, if the product AB is nil-clean then BA may not be nil-clean. Despite the fact that for many special cases, BA is also nil-clean, we finally found three counterexamples. All the way, the computer aid was decisive.

## 1 Introduction

An element $a$ in a unital ring $R$ is called *clean* if it is a sum of an idempotent and a unit, and, it is called *nil-clean* if it is a sum of an idempotent and a nilpotent. A nil-clean element is called *trivial* if the idempotent is 0 or 1. An element $a$ is *regular* if $a = axa$ for some $x$ and *unit-regular* if $x$ is a unit.

For any two elements $a, b$ in a unital ring $R$, $1 - ab$ *is a unit if and only if* $1 - ba$ *is a unit*. This result is known as Jacobson's lemma for units. It is known that Jacobson's lemma holds for Drazin invertible elements, for generalized Drazin invertible elements, for $\pi$-regular elements and unit-regular elements, but fails for clean elements. Moreover, Jacobson's lemma holds for strongly nil-clean elements and fails for nil-clean elements. An example in a *subring* of $\mathbb{M}_2(\mathbb{Z})$ was recently given in [5].

It is easy to see that, for nil-clean elements, the Jacobson lemma is equivalent to: *ab is nil-clean if and only if ba is nil-clean.*

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T. Y. Lam (private communication) asked whether a negative example could be found in the full matrix ring $\mathbb{M}_2(\mathbb{Z})$. This question turned out to be a very hard one, mainly because we do not know how to decompose nil-clean matrices into two factor products.

In this note, in section 2 we present our counterexamples. In section 3, we give the details of our final successful attempt to find a counterexample. All the way, the computer aid was decisive.

All the rings we consider are unital, PID means principal ideal domain. By $E_{11}$ we denote the $2 \times 2$ matrix with zero all entries, excepting the NW entry, which is 1.

## 2 The counterexamples

Recall that *over any PID, every $2 \times 2$ idempotent matrix is similar to $E_{11}$*. The following lemma will be useful.

**Lemma 1.** *Suppose that Jacobson's Lemma holds for $E_{11}$-nil-clean products $AB$. Then the Lemma holds in general.*

*Proof.* Indeed, if $AB = E + T$ and $U^{-1}EU = E_{11}$ then $(U^{-1}AU)(U^{-1}BU) = E_{11} + U^{-1}TU$ is $E_{11}$-nil-clean. By hypothesis, $(U^{-1}BU)(U^{-1}AU) = U^{-1}BAU$ is nil-clean and so is $BA$. $\square$

Further, recall the following characterization of nontrivial $2 \times 2$ integral nil-clean matrices (e.g. see [3]).

**Theorem 2.** *A $2 \times 2$ integral matrix $A$ is nontrivial nil-clean if and only if $A$ has the form $\begin{bmatrix} a+1 & b \\ c & -a \end{bmatrix}$ for some integers $a, b, c$ such that* $\det(A) \neq 0$ *and the system*

$$\begin{cases} x^2 + x + yz = 0 & (1) \\ (2a+1)x + cy + bz = a^2 + bc & (2) \end{cases}$$

*with unknowns $x, y, z$, has at least one solution over $\mathbb{Z}$. We can suppose $b \neq 0$ and if (2) holds, (1) is equivalent to*

$$bx^2 - (2a+1)xy - cy^2 + bx + (a^2 + bc)y = 0 \quad (3).$$

**Remark**. The equation (2) has the solution
(i) $(0, 0)$ if and only if $b$ divides $a^2$;
(ii) $(-1, 0)$ if and only if $b$ divides $(a+1)^2$;
(iii) $(a, b)$ if and only if $b$ divides $a^2 + a$.

All our counterexamples have the same $2 \times 2$ matrix $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$.

**(i)** $B = \begin{bmatrix} -27 & -47 \\ 14 & 28 \end{bmatrix}$. Then $BA = \begin{bmatrix} -195 & -343 \\ 112 & 196 \end{bmatrix}$ and $BA - E_{11}$ has zero trace and zero determinant. So it is square zero and $BA$ is $E_{11}$-nil-clean.

Next, using Theorem 2, we show that $AB = \begin{bmatrix} 2 & 18 \\ -11 & -1 \end{bmatrix}$ is not nil-clean. We have $a = 1$, $b = 18$ and $c = -11$ and clearly $b$ does not divide $a^2$, $(a+1)^2$ nor $a^2 + a$. So (according to the Remark after the theorem) the solutions $(0,0)$, $(-1,0)$, $(a,b) = (1,18)$ do not verify (2). The equation (3), $18x^2 - 3xy + 11y^2 + 18x - 197y = 0$, has only one more solution: $(-6,4)$ (see [1] or [4]). Now (2) is $3x - 11y + 18z = -197$; for $x = -6$, $y = 4$ we get $z = 7.5 \notin \mathbb{Z}$. Therefore $AB$ is *not* nil-clean.

**(ii)** $B = \begin{bmatrix} 17 & -37 \\ -13 & 26 \end{bmatrix}$. Then $BA = \begin{bmatrix} -77 & -117 \\ 52 & 78 \end{bmatrix}$ and $BA - E_{11}$ has zero trace and zero determinant. So it is square zero and $BA$ is $E_{11}$-nil-clean. Next, $AB = \begin{bmatrix} -18 & 30 \\ -14 & 19 \end{bmatrix}$, the same three solutions are eliminated and (3) has two more solutions: $(-6,4)$ and $(-17,20)$.

For the first, from (2) we get $z = 7.5 \notin \mathbb{Z}$ and for the second we obtain $z = 13.6 \notin \mathbb{Z}$. Hence $AB$ is *not* nil-clean.

**(iii)** $B = \begin{bmatrix} 11 & -25 \\ -9 & 18 \end{bmatrix}$. Then $BA = \begin{bmatrix} -53 & -81 \\ 36 & 54 \end{bmatrix}$ and $BA - E_{11}$ has zero trace and zero determinant. So it is square zero and $BA$ is $E_{11}$-nil-clean. Next, $AB = \begin{bmatrix} -14 & 22 \\ -12 & 15 \end{bmatrix}$, the same three solutions are eliminated and (3) has one more solution: $(-6,4)$.

From (2) we get $z = 7.5 \notin \mathbb{Z}$. Hence $AB$ is *not* nil-clean.

**(iv)** Here $AB = \begin{bmatrix} 100 & -94 \\ 105 & -99 \end{bmatrix} = \begin{bmatrix} -440 & 392 \\ -495 & 441 \end{bmatrix} + \begin{bmatrix} 540 & 486 \\ 600 & -540 \end{bmatrix}$ is nil-clean, so yields *no* counterexample.

## 3 How the counterexample was found

According to Lemma 1, to verify the failure, it suffices to show that if $AB$ is $E_{11}$-nil-clean then $BA$ might not be nil-clean.

This shows how a program which should (partly but not exhaustively) check this, should be designed.

By $z$ we denote the upper bound of the absolute value of the entries in the starting matrices.

Here is a *first set* of steps.

1) constructs two $2 \times 2$ integral matrices $A, B$;

2) multiplies $A$ (left) with $B$ (right);
3) subtracts 1 from the NW corner $[(AB)_{11} - 1]$;
4) takes the square of this matrix $[(AB - E_{11})^2]$;
5) if this square is $0_2$ it multiplies $B$ (left) with $A$ (right) and stores somewhere this product together with components; the pair will be called *valid*;
6) if this square is not $0_2$ the program discards this pair and continues, from 1), with another pair.

A *second set* of steps improved our search
a) eliminate the repetitions;
b) eliminate the idempotents and the nilpotents (which are obviously nil-clean);
c) eliminate all the units, that is, valid $BA$'s with determinant $\pm 1$; this is because trace 1 units are nil-clean (see [2]);
d) eliminate the initial pairs $A$, $B$ whenever $A$ or $B$ is a unit;
e) eliminate the initial pairs $A$, $B$ whenever $A$ or $B$ is diagonal.

Trying to find a counterexample, among others, we came to consider matrices of form $A = \begin{bmatrix} a & a+2 \\ a+1 & a+3 \end{bmatrix}$, which appear among the valid pairs given by computer. Since in the general case there was no solution at hand, we asked (the computer) for a $z = 5$ list of valid pairs.

Surprising, the corresponding matrix for $a = 2$, i.e. $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$ was missing from the list (but many slightly similar combinations were there, e.g. $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, $\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$). Indeed, here is a simple

**Proof**. Start with $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$. Then $AB - E_{11} = \begin{bmatrix} 2x+4z-1 & 2y+4t \\ 3x+5z & 3y+5t \end{bmatrix}$ must have zero trace and determinant. This gives $2x+3y+4z+5t = 1$ and $\det(AB)-(3y+5t) = 0$, i.e. $-2\det(B)-(3y+5t) = 0$. Hence $3y + 5t = -2\det(B) = 1 - 2(x + 2z)$, impossible.

Recall that $\det(AB - E_{11}) = \det(AB) - (AB)_{22} = \det(A)\det(B) - (3y + 5t) = -2\det(B) - (3y + 5t)$, where $(AB)_{22}$ denotes the SE entry of $AB$.

This reopened the hope of finding a counterexample because the matrix $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$ seemed to be somehow exceptional.

The proof above shows that for any $B \in \mathbb{M}_2(\mathbb{Z})$, $AB$ is not $E_{11}$-nil-clean. Of course, it was *too much to hope* that $AB$ *is not nil-clean for any* $B \in \mathbb{M}_2(\mathbb{Z})$, and this is not true, as we saw in Section 2, but maybe for a good choice of the matrix $B$,

(i) $BA$ is $E_{11}$-nil-clean and
(ii) $AB$ is not nil-clean.

Since $BA = \begin{bmatrix} 2x+3y & 4x+5y \\ 2z+3t & 4z+5t \end{bmatrix}$, for (i) we need $2x+3y+4z+5t = 1$ and $-2\det(B) - 4z - 5t = 0$.

Now we have $4z + 5t = -2\det(B) = 1 - 2x - 3y$ which is possible, with odd $y$ and even $t$.

We covered "by hand" (that is, using Theorem 2 and the remark after) some possible cases for $t \in \{0, \pm 2, \pm 4\}$ combined with $z \in \{0, \pm 1, \pm 2, \pm 3\}$, unsuccessfully (i.e. all $AB$'s were also nil-clean).

Therefore we finally "gave" this task to the computer.

The program for this was designed as follows.

Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$. We tried to check whether for a good choice of the matrix $B$,

(i) $BA$ is $E_{11}$-nil-clean and
(ii) $AB$ is not nil-clean.

Consequently, we browsed all $2 \times 2$ matrices $B$ (say with $z = 50$) and store all the $B$'s such that $(BA - E_{11})^2 = 0_2$ (this explain the term "initial" in the table below).

For these $B$, from the products $AB$ we subtract all nilpotents (say $b = 150$), i.e. the square zero matrices (with $b$ the the upper bound of the absolute value of the entries). We check $(AB - T)^2 = AB - T$, i.e. if $AB - T$ is idempotent.

If for some $T$, $AB - T$ is idempotent, we eliminate this $B$ and pass to the next $B$.

If the program finds a $B$ such that none of $AB - T$'s are idempotent, we have a *possible* (because of the bounds $z$ and $b$) counterexample.

If there is no counterexample, and $z$, $b$ are large enough, all $B$'s are finally eliminated.

For this procedure, large bounds $z$, $b$ can be covered in a reasonable computer time.

For $z = 50$ and $b = 150$ only four $B$'s remained. In the table below, we can see what happened for $z = 50$ and several values of $b$.

| $z$ | $b$ | cases not eliminated |
|---|---|---|
| 50 | initial | 43 |
| 50 | 20 | 19 |
| 50 | 40 | 10 |
| 50 | 80 | 6 |
| 50 | 150 | 4 |
| 50 | 200 | ... |

Since already for $b = 150$ the computer time was some 12 hours, we decided to deal directly (i.e. with Theorem 2 and the remark after) with these four matrices. Three of these are the desired counterexamples presented in section 2 and one of these has nil-clean $BA$.

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GRIGORE CĂLUGĂREANU,
Department of Mathematics,
Babeş-Bolyai University,
Str. Kogălniceanu 1, 400084 Cluj-Napoca, Romania.
Email: calu@math.ubbcluj.ro

Horia F. POP,
Department of Computer Science,
Babeş-Bolyai University,
Str. Kogălniceanu 1, 400084 Cluj-Napoca, Romania.
Email: hfpop@cs.ubbcluj.ro